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AVERAGING OF THE SYSTEM OF EQUATIONS OF MOTION OF A VISCOUS FLUID IN A POROUS MEDIUM†

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A system of equations of motion of a viscous fluid (with a small coefficient of viscosity) in a periodic porous medium is considered. The ratio of the period of the structure of the medium to the characteristic dimension of the problem is a small parameter. The pore size and period of the structure are of the same order of magnitude. A formal asymptotic solution is constructed and an averaged equation, which is an analogue of the Boussinesq equation, is derived.

We will adopt as a model of a porous system a continuous medium with periodically arranged cavities (see [1], for example). Let K be the cube $\{\xi \in \mathbb{R}^3 \mid 0 < \xi_j < 1, j = 1, 2, 3\}$ and let us place a certain set, consisting of a finite number of domains, in K and continue it periodically (with a period of unity) in the whole of the space R^3. Let A be the union of all the resulting sets. We assume that the boundary ∂A is a smooth manifold and the domain $\Omega = R^3\backslash A$ is connected, and that A_ε is the set obtained from A by a homothetic contraction ε^{-1} times (ε is a small parameter), that is, $A_\varepsilon = \{x \in R^3 \mid x/\varepsilon \in A\}$.

We will adopt as the geometric model of the pore system the domain $\Omega_\varepsilon = R^3\backslash A_\varepsilon$ and consider the system of equations of motion of a viscous fluid with a small coefficient of viscosity μ_ε in this set

$$
\begin{gathered}
\rho(\partial \upsilon/\partial t+(\upsilon,\nabla)\upsilon) = -\nabla p+\mu_\varepsilon \Delta\upsilon + f_\varepsilon(x/\varepsilon, x, t, \upsilon) \\
\partial\rho/\partial t + \operatorname{div}(\rho\upsilon) = 0, \quad \rho = Q(p), \quad \rho, \upsilon \in R^3 \\
\mu_\varepsilon = \varepsilon^\gamma \mu, \quad \gamma < 2, \quad \mu = \text{const}, \quad f_\varepsilon(\xi, x, t, \upsilon) = f(\xi, x, t\varepsilon^{2-\gamma}, \upsilon)
\end{gathered}
\tag{1}
$$

$x \in \Omega_\varepsilon$ and $t > 0$ (ρ is the density, υ is the velocity vector and p is the pressure), $f(\xi, x, \tau, \upsilon)$ and μ are independent of ε, the function f is 1-periodic with respect to $\xi \in \Omega$, and Q is a specified smooth function.

The no-slip condition

$$
\upsilon|_{\partial\Omega_\varepsilon} = 0 \tag{2}
$$

is imposed at the boundary of the domain Ω_ε and the initial values of the unknown functions

$$
t = 0: \quad \upsilon = 0, \quad p = 0 \tag{3}
$$

are specified.

The limiting behaviour of the solution of problem (1)–(3) as $\varepsilon \to 0$ is studied (a formal asymptotic solution is constructed).

There is an extensive literature ([2–8], for example) on the investigation of various equations defined in domains of this type. In particular, systems of Stokes and Navier–Stokes equations in porous media have been considered in [5–8] and averaged models of the Darcy law type [5–8], of the non-linear Darcy law type [8] and also the Brinkman law type [9] have been obtained. An averaged model of the Boussinesq equation type [10] will be obtained below for problem (1)–(3).

A formal asymptotic solution is sought in the form of functions of fast and slow variables

$$
\begin{gathered}
\upsilon = \varepsilon^{2-\gamma}(\bar{\upsilon}(x,\tau) + V(x/\varepsilon, x, \tau)) \\
p = \bar{p}(x,\tau) + \varepsilon P(x/\varepsilon, x, \tau), \quad \rho = \bar{\rho}(x,\tau) + \varepsilon R(x/\varepsilon, x, \tau) \\
\tau = t\varepsilon^{2-\gamma}, \quad \langle V \rangle = \int_{K\backslash A} V(\xi, x, \tau)\, d\xi = 0
\end{gathered}
\tag{4}
$$

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where $P(\xi, x, \tau)$, $V(\xi, x, \tau)$, $R(\xi, x, \tau)$ are 1-periodic functions of ξ.

Substituting (4) into (1) and using the formula for the differentiation of a complex function, we obtain

$$-\mu\Delta_\xi V+\nabla_x\bar{p}+\nabla_\xi P-f(\xi,x,\tau,\varepsilon^{2-\gamma}(\bar{\upsilon}+V))+\varepsilon^{3-2\gamma}\bar{\rho}(\bar{\upsilon}+V)\nabla_\xi V+$$

$$+\varepsilon^{2(2-\gamma)}(\bar{\rho}\partial(\bar{\upsilon}+V)/\partial\tau+R(\bar{\upsilon}+V)\nabla_\xi V+(\bar{\rho}+\varepsilon R)(\bar{\upsilon}+V)\nabla_x(\bar{\upsilon}+V))+ \tag{5}$$

$$+\varepsilon^{5-2\gamma}R\partial(\bar{\upsilon}+V)/\partial\tau+\varepsilon(\nabla_x p-2\mu\Delta_{x\xi}V)-\varepsilon^2\mu\Delta_x(\bar{\upsilon}+V)-h\bar{\upsilon}-\tilde{h}(x,\tau)=0$$

$$\varepsilon^{-1}\operatorname{div}_\xi(\bar{\rho}V)+\partial\bar{\rho}/\partial\tau+\operatorname{div}_x(\bar{\rho}(\bar{\upsilon}+V))+\operatorname{div}_\xi(R(\bar{\upsilon}+V))+\varepsilon(\partial R/\partial\tau+\operatorname{div}_x(R(\bar{\upsilon}+V))) \tag{6}$$

$$\bar{\rho}-Q(\bar{p})+\varepsilon R-(Q(\bar{p}+\varepsilon P)-Q(\bar{p}))=0 \tag{7}$$

$$\Delta_{x\xi}=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i\partial\xi_i}$$

h is a constant 3×3 matrix, and $\tilde{h}(x,\tau)$ is a vector function, chosen from the condition $\langle V\rangle = 0$.

On the boundary $\partial\Omega_\varepsilon$

$$\bar{\upsilon}+V=0 \tag{8}$$

The functions $\bar{p}$, $\bar{\upsilon}$, $\bar{\rho}$, $\tilde{h}$, P, V, R are sought in the form of regular series in the parameters $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^{3-2\gamma}$, $\varepsilon_3 = \varepsilon^{2-\gamma}$.

Substituting these series into Eqs (5)–(8), we obtain

$$\sum_{k,l,m=0}^{\infty}\varepsilon_1^k\varepsilon_2^l\varepsilon_3^m(\nabla_x\bar{p}_{klm}+h\bar{\upsilon}_{klm}+\tilde{h}_{klm})+\sum_{k,l,m=0}^{\infty}\varepsilon_1^k\varepsilon_2^l\varepsilon_3^m(-\mu\Delta_\xi V_{klm}+\nabla_\xi P_{klm})=0$$

$$\varepsilon^{-1}\sum_{k,l,m=0}^{\infty}\varepsilon_1^k\varepsilon_2^l\varepsilon_3^m(\operatorname{div}_\xi(\bar{\rho}_{000}V_{klm})-\varphi^1_{klm})=0$$

$$\bar{\rho}_{000}-Q(\bar{p}_{000})+\sum_{\substack{k,l,m=0\\(k,l,m)\neq(0,0,0)}}^{\infty}\varepsilon_1^k\varepsilon_2^l\varepsilon_3^m((R_{klm}-\left.\frac{\partial Q}{\partial p}\right|_{p=\bar{p}_{000}}P_{klm}-\varphi^3_{klm})+$$

$$+(\bar{\rho}_{k+1,l,m}-\left.\frac{\partial Q}{\partial p}\right|_{p=\bar{p}_{000}}P_{k+1,l,m})=0 \tag{9}$$

$$V_{klm}=-\upsilon_{klm},\quad \xi\in\partial\Omega$$

The three subscripts correspond to the number of the coefficient in the regular expansion in powers of ε_1, ε_2, ε_3: φ^j_{klm} depends on $\bar{\rho}_{qsr}$, $\bar{\upsilon}_{qsr}$, $\bar{p}_{qsr}$, R_{qsr}, V_{qsr}, P_{qsr} with the subscripts q, s and r such that $q \leqslant k$, $s \leqslant l$, $r \leqslant m$, where just one of these inequalities is strict and φ^3_{klm} also depend on $\bar{p}_{klm}$.

Note that

$$\langle\varphi^1_{0lm}\rangle=0,\quad \langle\varphi^1_{k+1,l,m}\rangle=\partial\bar{\rho}_{klm}/\partial\tau+\operatorname{div}_x(\bar{\rho}_{klm}\bar{\upsilon}_{000}+\bar{\rho}_{000}\bar{\upsilon}_{klm})+$$

$$+\langle\operatorname{div}_x(\bar{\rho}_{klm}V_{000})+\operatorname{div}_x(\bar{\rho}_{000}V_{klm})\rangle+\bar{\psi}_{klm}(x,\tau) \tag{10}$$

where $\bar{\psi}_{klm}(x, \tau)$ also depend on $\bar{\rho}_{qsr}$, $\bar{\upsilon}_{qsr}$ with the subscripts q, s, and r which satisfy the same inequalities as above, $(k, l, m) \neq (0, 0, 0)$.

The matrix h and the vector $\tilde{h}_{klm}$ are chosen from the condition

$$\langle V_{klm}\rangle = 0 \tag{11}$$

The coefficients $\bar{p}_{klm}$, $\bar{\upsilon}_{klm}$, $\bar{\rho}_{klm}$, $\tilde{h}_{klm}$, V_{klm}, R_{klm} are constructed according to the following algorithm. Let $W(\xi)$ be a 3×3 matrix function and let $F(\xi)$ be a three-element row matrix, 1-periodic with respect to ξ, and a solution of the problem

$$-\mu\Delta_\xi W+\nabla_\xi F=E,\quad \operatorname{div}_\xi W=0,\quad \xi\in\Omega$$
$$W=0;\quad \xi\in\partial\Omega \tag{12}$$

E is the unit 3×3 matrix. Let us put

$$V_{klm} = W(h\bar{\upsilon}_{klm} + \tilde{h}_{klm}) + \tilde{V}_{klm} - \bar{\upsilon}_{klm} \tag{13}$$

where h and $\tilde{h}_{klm}$ are chosen from the condition that $\langle V_{klm}\rangle = 0$, that is, $(V_{klm}, P_{klm}, R_{klm})$ is a solution of the problem

$$-\mu\Delta_\xi V_{klm} + \nabla_\xi P_{klm} = \varphi^2_{klm}, \quad \xi \in \Omega \tag{14}$$

$$\mathrm{div}_\xi(\rho_{000} V_{klm}) = \varphi^1_{klm}, \quad \xi \in \Omega \tag{15}$$

$$R_{klm} = \left.\frac{\partial Q}{\partial p}\right|_{p=\bar{p}_{000}} P_{klm} + \varphi^3_{klm}, \quad \xi \in \Omega \tag{16}$$

$$\tilde{V}_{klm} = 0, \quad \xi \in \partial\Omega \tag{17}$$

$$h = \langle W\rangle^{-1}, \quad \tilde{h}_{klm} = -\langle W\rangle^{-1}\langle \tilde{V}_{klm}\rangle \tag{18}$$

Actually, it follows from (13) and (18) that

$$\langle W(h\bar{\upsilon}_{klm} + \tilde{h}_{klm}) + \tilde{V}_{klm} - \bar{\upsilon}_{klm}\rangle = 0$$

The condition for problem (14), (15), (17) to be solvable [2, p. 167] is

$$\langle \varphi^1_{k+1lm}\rangle = 0$$

and relations (9)–(11) yield the equations for $\bar{\rho}_{klm}, \bar{p}_{klm}, \bar{\upsilon}_{klm}$

$$h\bar{\upsilon}_{klm} + \tilde{h}_{klm} + \nabla_x \bar{p}_{klm} = 0 \tag{19}$$

$$\begin{gathered}
\partial\bar{\rho}_{klm}/\partial\tau + \mathrm{div}_x(\bar{\rho}_{klm}\bar{\upsilon}_{000} + \bar{\rho}_{000}\bar{\upsilon}_{klm}) + \bar{\psi}_{klm}(x,\tau) = 0 \\
\bar{\rho}_{klm} = \left.\frac{\partial Q}{\partial p}\right|_{p=p_{000}} \bar{p}_{klm}, (k,l,m) \neq (0,0,0) \\
\langle W\rangle^{-1}\bar{\upsilon}_{000} + \nabla\bar{p}_{000} - \langle W\rangle^{-1}\langle V_{000}\rangle = 0 \\
\partial\bar{\rho}_{000}/\partial\tau + \mathrm{div}_x(\bar{\rho}_{000}\bar{\upsilon}_{000}) = 0, \quad \bar{\rho}_{000} = Q(\bar{p}_{000})
\end{gathered} \tag{20}$$

with the homogeneous initial conditions $t = 0$: $\bar{p}_{klm} = 0$, $\bar{\upsilon}_{klm} = 0$.

System (20) is an averaged system of zeroth-order equations. Substituting υ_{000} from the first equation and ρ_{000} from the third equation into the second equation, we obtain

$$\partial Q(\bar{p}_{000})/\partial\tau - \mathrm{div}_x(Q(\bar{p}_{000})\langle W\rangle(\nabla\bar{p}_{000} - \langle W\rangle^{-1}\langle\tilde{V}_{000}\rangle)) = 0 \tag{21}$$

where $\tilde{V}_{000}$ is the solution of the problem

$$\begin{gathered}
-\mu\Delta_\xi V_{000} + \nabla_\xi P_{000} = f(\xi, x, \tau, 0), \quad \xi \in \Omega \\
\mathrm{div}_\xi V_{000} = 0, \quad \xi \in \Omega; \quad V_{000}|_{\partial\Omega} = 0
\end{gathered} \tag{22}$$

Equations (14)–(20) have to be solved in the following order: problem (14), (15), (17) is first solved for the pair $\tilde{V}_{klm}, P_{klm}$ for each fixed set of k, l, m (when $(k, l, m) = (0, 0, 0)$, it is problem (22)). We next determine $\tilde{h}_{klm}$ from (18) and then solve problems (19), (20) and, finally, R_{klm} is determined from (16).

We note that the averaged model (21), (22) is an analogue of the Boussinesq model: it becomes the Boussinesq model [10] in the case of a linear dependence of $Q(p)$. If $f(\xi, x, \tau, \upsilon) = 0$ when $\tau \in [0, \tau_0]$, $\tau_0 > 0$, it can be shown by induction that $\tilde{V}_{klm} = 0$ when $\tau \in (0, \tau_0)$ so that $\bar{\upsilon} + V$ are asymptotically equal to zero when $t = 0$. Conditions (1)–(3) are exactly satisfied asymptotically.

By analogy with problem (1) in the domain Ω_ε, problem (1)–(3) may be treated in the domain $G_\varepsilon \times R$, where $G_\varepsilon = \{x' \in R^2, x'/\varepsilon \in G_1\}$, the domain G_1 is bounded in R^2 with a piecewise-smooth boundary, $x' = (x_1, x_2)$, the function f_ε is solely dependent on x'/ϵ, x_3, t, υ and its first two components $f_{\varepsilon 1}, f_{\varepsilon 2}$ are zero.

This model simulates the motion of a viscous fluid in a channel of complex form. In constructing the asymptotic form we use (4) with x/ε replaced by x'/ε, x replaced by x_3 and $\bar{\upsilon} = (0, 0, \bar{\upsilon}_3)^*$. The analogue of the operator for the mean $\langle\cdot\rangle$ is the integral $\int_{G_1} \cdot\, \delta\xi'$, $\xi' = (\xi_1, \xi_2)$ (the third component of the vectors $\langle V_{klm}\rangle$ is equal to zero).

The procedure which has been described above as applied to problem (1)–(3), in the first approximation yields

$$\frac{\partial Q(\bar{p}_{000})}{\partial \tau} - \frac{\partial}{\partial x_3}(Q(\bar{p}_{000})\langle \hat{W}\rangle\left(\frac{\partial \bar{p}_{000}}{\partial x_3} - \langle \hat{W}\rangle^{-1}\langle \hat{V}_{000}\rangle\right) = 0, \quad \bar{p}_{000}|_{t=0} = 0$$

in the domain $G_\varepsilon \times R$, where $\hat{W}(\xi')$ is the solution of Poisson's equation

$$-\mu\Delta_{\xi'}\hat{W} = 1, \quad \xi' \in G_1; \quad \hat{W}|_{\partial G_1} = 0$$

and $\hat{V}_{000}(\xi')$ is the solution of Poisson's equation

$$-\mu\Delta_{\xi'}\hat{V}_{000} = f^3(\xi', x_3, \tau, 0), \quad \xi' \in G_1, \quad \hat{V}_{000}|_{\partial G_1} = 0$$

Hence the homogenization procedure in the zeroth approximation yields an analogue of the Boussinesq equation in the case of an unsteady-state system of equations of motion for a viscous fluid with a small coefficient of viscosity.

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